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# On the Divergence Phenomenon in Hermite–Fejér Interpolation

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Generalizing results of L. Brutman and I. Gopengauz (1999, *Constr. Approx.* **15**, 611–617), we show that for any nonconstant entire function $f$ and any interpolation scheme on $[-1, 1]$, the associated Hermite–Fejér interpolating polynomials diverge on any infinite subset of $\mathbb{C}\backslash[-1, 1]$. Moreover, it turns out that even for the locally uniform convergence on the open interval $]-1, 1[$ it is necessary that the interpolation scheme converges to the arcsine distribution. 

## 1. INTRODUCTION

The classical field of Hermite–Fejér interpolation, which is devoted to polynomials $H_{2n-1} \in \mathscr{P}_{2n-1}$ satisfying for a given function $f$ on $[-1, 1]$ in $n$ interpolation nodes $-1 \leqslant x_1 < \cdots < x_n \leqslant 1$ the Hermite-type conditions

$$\begin{aligned} H_{2n-1}(x) &= f(x_i), \qquad & i = 1, ..., n, \\ H'_{2n-1}(x_i) &= 0, \qquad & i = 1, ..., n, \end{aligned} \tag{1}$$

and its generalizations have attracted the attention of many mathematicians (see [5] for an extensive bibliography). Here, the oldest and by far most celebrated result is due to Fejér [3], who proved that for each continuous function $f$, the Hermite–Fejér interpolants in the zeros of the $n$th Chebyshev polynomials converge to $f$ uniformly on $[-1, 1]$, which is in striking contrast to the negative result in Faber's theorem concerning Lagrange interpolation.

A large number of papers is mainly devoted to finding conditions for the convergence in the case of special interpolation schemes as, for instance, the zeros of Jacobi polynomials.

Recently, in an interesting article, Brutman and Gopengauz [2] discussed for arbitrary interpolation nodes the divergence of Hermite–Fejér



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interpolants to the special function $f_0(z)=z$ in the complex plan, thus pointing out the difference to the Lagrange interpolation process, which is well known to converge locally uniformly in $\mathbb{C}$.

Their paper is apparently also inspired by the surprising result of Berman [1] in which pointwise divergence for $f_0(z)=z$ holds in $[-1,1]\backslash\{-1,0,1\}$, if the underlying interpolation scheme is given by equidistant points.

Though the proofs in [2] are relatively elementary, at some point complex potential theory, which already plays a significant role in other fields of polynomial and rational approximation, is needed.

One purpose of this paper is to show how potential theory can be used to formulate and prove results in the theory of Hermite–Fejér interpolation. Extending the result of Brutman and Gopengauz it will be shown that for any nonconstant entire function and each interpolation scheme, the Hermite–Fejér interpolation process diverges outside the interval $[-1,1]$ except for, roughly speaking (see Theorem 3), at most a finite number of points.

In addition, it will be shown that in the case of nonconstant entire functions even locally uniform convergence in $]-1,1[$ is impossible, unless the interpolation scheme converges to the equilibrium measure (i.e., the arcsine distribution) on $[-1,1]$ in the weak-star sense.

The corresponding precise statement in Theorem 4 gives an appropriate interpretation of the aforementioned divergence result of Berman, which then no longer appears that surprising.

## 2. NOTATION FROM POTENTIAL THEORY

We need to introduce some notation from potential theory (for more details, the reader is referred to the recent monograph of Saff and Totik [8]).

For a unit (Borel-) measure $\mu$ on $[-1,1]$ we define its logarithmic potential via

$$U^{\mu}(x) := \int \log \frac{1}{|x-y|}\, d\mu(y) \qquad (x\in\mathbb{C}),$$

which is a function superharmonic in $\mathbb{C}$ and harmonic outside its support $\mathrm{supp}(\mu)$. In addition, set

$$\begin{aligned} &E(\lambda,\mu) := \{z\in\mathbb{C} : U^{\mu}(z) > \lambda\} \qquad (\lambda\in\mathbb{R}) \qquad \text{and} \\ &\lambda(\mu) := \inf_{x\in[-1,1]} U^{\mu}(x). \end{aligned} \tag{2}$$

By the lower semicontinuity of $U^\mu$, for each $-\infty < \lambda < \lambda(\mu)$, $E(\lambda, \mu)$ is an open neighbourhood of $[-1, 1]$.

It is well known that $U^\mu$ is constant on $[-1, 1]$ if and only if $\mu$ is the so-called Robin equilibrium measure (or arcsine distribution) on $[-1, 1]$; i.e.,

$$d\mu(x) = \frac{1}{\pi} \frac{dx}{\sqrt{1-x^2}} \qquad (x \in ]-1, 1[).$$

By an interpolation scheme $X = (x_i^{(n)})$ in $[-1, 1]$ we understand a triangular matrix of interpolation nodes $-1 \leqslant x_1^{(n)} < \cdots < x_n^{(n)} \leqslant 1$, $n \geqslant 1$.

We say that along some subsequence $\Lambda \subset \mathbb{N}$ the interpolation scheme $(x_i^{(n)})$ has weak-star limit $\mu^*$, if along $\Lambda$ the unit measures $\nu_n$ associating with each point $x_i^{(n)}$ the equal mass $1/n$ converge to (the unit measure on $[-1, 1]$) $\mu^*$ in the weak-star sense. Then, as is well known,

$$\log \left| \prod_{i=1}^{n} (x - x_i^{(n)}) \right|^{1/n} = U^{\nu_n}(x) \xrightarrow{n \in \Lambda} U^{\mu^*}(x)$$

locally uniformly for $x \in \mathbb{C} \backslash [-1, 1]$.

Finally, for a set $A \subset \mathbb{C}$ denote by $\|\cdot\|_A$ the Chebyshev norm on $A$.

## 3. AN AUXILIARY INTERPOLATION PROBLEM

Suppose $h$ is a function continuous on $[-1, 1]$. For an interpolation scheme $X = (x_i^{(n)})$ in $[-1, 1]$, denote by $r_{2n-1} = r_{2n-1}(h; X; \cdot) \in \mathscr{P}_{2n-1}$ the polynomial satisfying the Hermite-type interpolation conditions

$$\begin{aligned} r_{2n-1}(x_i^{(n)}) &= 0, \qquad & i &= 1, ..., n, \\ r'_{2n-1}(x_i^{(n)}) &= h(x_i^{(n)}), \qquad & i &= 1, ..., n. \end{aligned} \tag{3}$$

By the Hermite interpolation formula,

$$r_{2n-1}(x) = \sum_{i=1}^{n} h(x_i^{(n)}) \frac{(\omega_n(x))^2}{(\omega_n'(x_i^{(n)}))^2} \frac{1}{(x - x_i^{(n)})}, \tag{4}$$

where $\omega_n(x) := \prod_{k=1}^{n} (x - x_k^{(n)})$.

THEOREM 1. *Suppose h has exactly* $0 \leqslant s < \infty$ *zeros in* $[-1, 1]$. *Le* $\Lambda$ *be a subsequence of* $\mathbb{N}$. *Then either*

(i) *for every set* $A \subset \mathbb{C} \backslash [-1, 1]$ *consisting of at least* $s+1$ *points, there holds*

$$\|r_{2n-1}\|_A \xrightarrow{n \in \Lambda} \infty,$$

*or*

(ii) *there exists a subsequence* $\Lambda_1 \subset \Lambda$, *a weak-star limit* $\mu^*$ *along* $\Lambda_1$ *of the interpolation scheme, and a point* $t_0$ *in the support of* $\mu^*$ *such that for some disk* $D(t_0)$ *centered at* $t_0$,

$$\|r_{2n-1}\|_{D(t_0)} \xrightarrow{n \in \Lambda_1} 0.$$

*In addition, if* (ii) *holds, one may choose any point* $t_0$ *for which* $U^{\mu^*}(t_0) = \infty$ (*if such a point exists*), *e.g., if* $\mu^*(\{t_0\}) > 0$.

*Remark.* It can be shown that in the most interesting cases (ii) is impossible, for instance

(a) if each weak-star limit of the interpolation scheme is a measure, the possible point masses of which are not zeros of $h$ (in particular, if any weak-star limit is a continuous measure on $[-1, 1]$);

(b) if $h$ is entire or only analytic in a sufficiently large neighbourhood of $[-1, 1]$, e.g., if $h$ is analytic in a neighbourhood of the set $\{z \in \mathbb{C}: \operatorname{dist}(z, [-1, 1]) \leqslant 2\}$ (see the reasoning in the proof of Theorem 3 and [4, p. 64–65]).

That (ii) in Theorem 1 is possible can be seen from the following

EXAMPLE. Look at the $\mathscr{C}^\infty$-function $h(x) = \exp(-1/x^2)$ and consider the interpolation scheme $x_i^{(n)} = y_i^{(n)}/n$, where $y_1^{(n)}, ..., y_n^{(n)}$ are the zeros of the $n$th Chebyshev polynomial $T_n$ on $[-1, 1]$, normalized by $\|T_n\|_{[-1,1]} = 1$. Performing a linear transformation, it is easy to see that

$$r_{2n-1}(x) = \sum_{i=1}^{n} h(x_i^{(n)}) \frac{1}{n^2} \frac{(T_n(nx))^2}{(T_n'(y_i^{(n)}))^2} \frac{1}{(x - x_i^{(n)})}.$$

Since the Green function $g(\cdot, \infty)$ of the complement of $[-1, 1]$ has a logarithmic pole at $\infty$, there exists $n_0 \geqslant 2$ such that for $|x| = n \geqslant n_0$, $g(nx, \infty) \leqslant 3 \log n$. Taking into account that $|T_n'(y_i^{(n)})| \geqslant n$ (see [7, p. 7]), the Bernstein–Walsh lemma (see [8, p. 153]) thus implies that if $|x| = n \geqslant n_0$, then

$$|r_{2n-1}(x)| \leqslant \frac{1}{n^2} h\left(\frac{1}{n}\right) \frac{|T_n(nx)|^2}{\operatorname{dist}(nx, [-1, 1])}$$
$$\leqslant h\left(\frac{1}{n}\right) \exp(2ng(nx, \infty)) \leqslant h\left(\frac{1}{n}\right) \exp(6n \log n).$$

By the maximum principle, this estimate also holds for $|x| \leqslant n$. Inserting $h(1/n) = \exp(-n^2)$ yields that $r_{2n-1} \to 0$ locally uniformly in $\mathbb{C}$.

*Proof of Theorem* 1. Assume contrary to (i) that there exists a set $A \subset \mathbb{C} \setminus [-1, 1]$ consisting of $s+1$ points such that for some subsequence $\Lambda_0 \subset \Lambda$, $\|r_{2n-1}\|_A$ remains bounded along $\Lambda_0$.

Now, $r_{2n-1}(x) = \omega_n^2(x)\, q_n(x)$, where

$$q_n(x) := \sum_{k=1}^{n} \frac{h(x_k^{(n)})}{(\omega_n'(x_k^{(n)}))^2} \frac{1}{(x - x_k^{(n)})}$$

is a rational function of degree $\leqslant n$. If for some index $k$ we have sign $h(x_k^{(n)}) = \operatorname{sign} h(x_{k+1}^{(n)}) \neq 0$, then $q_n$ has at least one zero in $]x_k^{(n)}, x_{k+1}^{(n)}[$. Therefore, $q_n$ has at least $\max(n - s - s_n' - 1, 0)$ zeros of this kind, interlacing with the points $x_k^{(n)}$, where $0 \leqslant s_n' \leqslant s$ denotes the number of points $x_l^{(n)}$ for which $h(x_l^{(n)}) = 0$. But if $h(x_l^{(n)}) = 0$, then $r_{2n-1}$ has a double zero at $x_l^{(n)}$. Thus, $r_{2n-1}$ has at most $s$ zeros outside the interval $[-1, 1]$, and we denote them by $\zeta_1^{(n)}, ..., \zeta_{s_n}^{(n)}$, $0 \leqslant s_n \leqslant s$.

Set $\tilde{\omega}_n(x) := \prod (x - \eta)$, where the product is taken over the zeros $\eta$ of $r_{2n-1}$ in $[-1, 1]$, counted with multiplicities. Then for some real coefficient $\alpha_{2n-1}$,

$$r_{2n-1}(x) = \alpha_{2n-1} \tilde{\omega}_n(x) \prod_{k=1}^{s_n} (x - \zeta_k^{(n)}).$$

By Helly's selection theorem (see [8, p. 3]), there exists a subsequence $\Lambda_1 \subset \Lambda_0$ and a unit measure $\mu^*$ on $[-1, 1]$ such that along $\Lambda_1$, the normalized zero counting measures associated with the polynomial $\tilde{\omega}_n$ converge to $\mu^*$ in the weak-star sense. Consequently,

$$\frac{1}{2n} \log \frac{1}{|\tilde{\omega}_n(x)|} \xrightarrow{n \in \Lambda_1} U^{\mu^*}(x) \tag{5}$$

locally uniformly for $x \in \mathbb{C} \setminus [-1, 1]$.

Since $s_n \leqslant s < \# A$, we may (after possibly passing to another subsequence) w.l.o.g. assume that there exists a point $x_0 \in A$ such that for some constant $c_0$,

$$|\zeta_i^{(n)} - x_0| \geqslant c_0 > 0 \qquad (n \in \Lambda_1,\ i = 1, ..., s_n). \tag{6}$$

Next, by the maximum principle (see [8, Remark 1.1.6]), there exist $t_0 \in \operatorname{supp}(\mu^*)$ and $\varepsilon > 0$ such that $U^{\mu^*}(t_0) > U^{\mu^*}(x_0) + 4\varepsilon$. Since $U^{\mu^*}$ is lower semicontinuous,

$$U^{\mu^*}(x) > U^{\mu^*}(x_0) + 4\varepsilon \qquad (x \in D_0), \tag{7}$$

where $D_0$ is some open disk centered at $t_0$.

Now, let $I \subset [-1, 1] \cap D_0$ be a closed, nondegenerate interval containing $t_0$. Consider the segments $I_\delta := \{x + i\delta : x \in I\}$, $0 \leqslant \delta \leqslant \delta_0$, where $\delta_0 > 0$ is such that $I_{\delta_0} \subset D_0$. In addition, denote by $g_\delta(\cdot, \infty) = g_{\mathbb{C} \setminus I_\delta}(\cdot, \infty)$ the Green function of the complement of $I_\delta$ with pole at infinity.

Since the Green function is continuous in $\mathbb{C}$ and vanishes on $I_\delta$, we may choose the parameter $0 < \delta_1 = \delta_1(\varepsilon) \leqslant \delta_0$ so small that for $x$ in a neighbourhood of $I$,

$$g_{\delta_1}(x, \infty) \leqslant \varepsilon.$$

Thus, by the Bernstein–Walsh lemma,

$$|\tilde{\omega}_n(x)| \leqslant \|\tilde{\omega}_n\|_{I_{\delta_1}} \exp(2n\varepsilon) \tag{8}$$

for $x$ in a neighborhood of $I$.

But by (5) and (7) for $n \in \Lambda_1$ sufficiently large,

$$\begin{aligned} |\tilde{\omega}_n(y)|^{1/(2n)} &\leqslant \exp(-U^{\mu^*}(y) + \varepsilon) \\ &\leqslant |\tilde{\omega}_n(x_0)|^{1/(2n)} \exp(U^{\mu^*}(x_0) - U^{\mu^*}(y) + 2\varepsilon) \\ &\leqslant |\tilde{\omega}_n(x_0)|^{1/(2n)} \exp(-2\varepsilon) \qquad (y \in I_{\delta_1}), \end{aligned}$$

so that by virtue of (8) for such $n$ and for $x$ in a neighbourhood of $t_0 \in I$,

$$|\tilde{\omega}_n(x)| \leqslant |\tilde{\omega}_n(x_0)| \exp(-2n\varepsilon).$$

This implies that for $n \in \Lambda_1$ sufficiently large and $x$ in this neighbourhood of $t_0$,

$$
\begin{aligned}
|r_{2n-1}(x)| &\leqslant |\alpha_{2n+1}|\, |\tilde{\omega}_n(x_0)| \exp(-2n\varepsilon) \left| \prod_{k=1}^{s_n} (x - \zeta_k^{(n)}) \right| \\
&= |r_{2n-1}(x_0)| \exp(-2n\varepsilon) \prod_{k=1}^{s_n} \frac{|x - \zeta_k^{(n)}|}{|x_0 - \zeta_k^{(n)}|} \\
&\leqslant |r_{2n-1}(x_0)| \exp(-2n\varepsilon)\, c_1 \qquad (9)
\end{aligned}
$$

with a constant $c_1$ not depending on $n$ (but on $s$ and the constant $c_0$ in (6)). In particular, since $|r_{2n-1}(x_0)|$ is bounded, $r_{2n-1}$ has to converge to 0 along $\Lambda_1$ uniformly in a neighbourhood of $t_0$. Thus, (ii) holds, and the proof of Theorem 1 is complete. ∎

THEOREM 2. *Let $h$ be as in Theorem* 1. *Suppose that along some subsequence $\Lambda \subset \mathbb{N}$ the interpolation scheme has weak-star limit $\mu^*$. Then at least one of the following assertions holds*:

(i) *For each* (*nondegenerate*) *closed subinterval $I \subset [-1, 1]$ on which the logarithmic potential $U^{\mu^*}$ is not identically equal to the constant* $\sup\{U^{\mu^*}(\zeta) \colon \zeta \in \mathbb{C}\}$,

$$\|r_{2n-1}\|_I \xrightarrow{n \in \Lambda} \infty.$$

(ii) *There exists a subsequence $\Lambda_1 \subset \Lambda$ and a point $t_0 \in \operatorname{supp}(\mu^*)$ such that for some disk $D(t_0)$ centered at $t_0$*,

$$\|r_{2n-1}\|_{D(t_0)} \xrightarrow{n \in \Lambda_1} 0.$$

*In addition, we may choose in* (ii) *any point $t_0$ satisfying $U^{\mu^*}(t_0) = \infty$* (*if such a point exists*).

*Proof.* Suppose (i) does not hold. Then there exists a nondegenerate closed subinterval $I$ of $[-1, 1]$ on which $U^{\mu^*}$ is not identically equal to its global supremum and, moreover, for some subsequence $\Lambda_1 \subset \Lambda$,

$$\sup_{n \in \Lambda_1} \|r_{2n-1}\|_I < \infty.$$

By the lower semicontinuity of the potential $U^{\mu^*}$, there exists $x_0 \in I$ with the property that $U^{\mu^*}(x_0) = \inf_I U^{\mu^*}$. Choose $t_0 \in [-1, 1]$ and $\varepsilon > 0$ such that $U^{\mu^*}(t_0) > U^{\mu^*}(x_0) + 5\varepsilon$. Of course we may select any point satisfying $U^{\mu^*}(t_0) = \infty$ (if such a point exists).

Again, since $U^{\mu^*}$ is lower semicontinuous, we can choose an open disk $D_0$ centered at $t_0$ such that

$$U^{\mu^*}(t) > U^{\mu^*}(x_0) + 5\varepsilon \qquad (t \in D_0). \qquad (10)$$

As in the proof of Theorem 1 we write

$$r_{2n-1}(x)=\alpha_{2n-1}\tilde{\omega}_n(x)\prod_{i=1}^{s_n}(x-\zeta_i^{(n)}),$$

where the $\zeta_j^{(n)}$ are the finitely many zeros of $r_{2n-1}$ outside the interval $[-1,1]$. As before, denote by $g_0(\cdot,\infty)$ the Green function of the complement of $I$ with pole at $\infty$.

From the mean-value inequality property

$$U^{\mu^*}(x_0)\geqslant\frac{1}{\pi r^2}\int_{|\zeta-x_0|\leqslant r}U^{\mu^*}(\zeta)\,dm_2(\zeta)\qquad(r>0)$$

with respect to the two-dimensional Lebesgue volume $m_2$ we deduce that for every neighbourhood $U$ of $x_0$ the planar Lebesgue measure of the set $\{x\in U: U^{\mu^*}(x)\leqslant U^{\mu^*}(x_0)\}$ is positive. Therefore, we may find a positive constant $c_0$ and a sequence of points $z_n$, $n\in\Lambda$, with the following properties:

(a) $g_0(z_n,\infty)\leqslant\varepsilon$ (i.e., the $z_n$ are sufficiently close to $I$),
(b) $U^{\mu^*}(z_n)\leqslant U^{\mu^*}(x_0)$,
(c) $\operatorname{dist}(z_n,[-1,1])\geqslant c_0$,
(d) $|z_n-\zeta_i^{(n)}|\geqslant c_0$.

Since $|\tilde{\omega}_n|^{1/2n}\to\exp(-U^{\mu^*})$ locally uniformly in $\mathbb{C}\setminus[-1,1]$ along $\Lambda$ and taking into account property (c) as well as (b) and (10), we obtain that for $t\in D_0$ and $n\in\Lambda$ sufficiently large,

$$\exp(-U^{\mu^*}(t))\leqslant\exp(-U^{\mu^*}(z_n)-5\varepsilon)\leqslant|\tilde{\omega}_n(z_n)|^{1/(2n)}\exp(-4\varepsilon).$$

As in the proof of Theorem 1 it follows that for $t$ in a neighbourhood of $t_0$ and $n\in\Lambda$ sufficiently large,

$$|\tilde{\omega}_n(t)|^{1/(2n)}\leqslant|\tilde{\omega}_n(z_n)|^{1/(2n)}\exp(-2\varepsilon).$$

Thus, for such $t$ and $n$,

$$|r_{2n-1}(t)|\leqslant|r_{2n-1}(z_n)|\exp(-2n2\varepsilon)\,c_1\leqslant\|r_{2n-1}\|_I\exp(-2n\varepsilon)\,c_1,$$

where $c_1$ is independent of $n$ (see (d) and the reasoning in (9) in the proof of Theorem 1) and where the last inequality follows from (a) and the Bernstein–Walsh lemma.

But $\|r_{2n-1}\|_I$ is assumed to be bounded along $\Lambda_1$ so that $|r_{2n-1}|\xrightarrow{n\in\Lambda_1}0$ uniformly in a neighbourhood of $t_0$; i.e., (ii) holds. ∎

## 4. DIVERGENCE OF HERMITE–FEJÉR INTERPOLANTS

Let $f$ be continuous on $[-1, 1]$ and denote by $H_{2n-1} = H_{2n-1}(f; X; \cdot) \in \mathscr{P}_{2n-1}$ the Hermite–Fejér interpolants (see (1)) associated with an interpolation scheme $X = (x_i^{(n)})$. It is easy to see that by the Montel theorem the Hermite–Fejér interpolants to a nonconstant entire (or only analytic on $[-1, 1]$) function have to be unbounded in each neighbourhood of $[-1, 1]$. But one can say more:

THEOREM 3. *Let $f$ be a nonconstant entire function and, say, $f'$ has exactly $s$ zeros in $[-1, 1]$. Then for any set $A \subset \mathbb{C} \backslash [-1, 1]$ consisting of at least $s+1$ points there holds*

$$\lim_{n \to \infty} \|f - H_{2n-1}\|_A = \infty.$$

*Proof.* It is well known [4, p. 64] that the Hermite interpolants $\tilde{H}_{2n-1} \in \mathscr{P}_{2n-1}$ interpolating $f$ and the derivative of $f$ in the points $x_i^{(n)}$ converge locally uniformly in $\mathbb{C}$ to the function $f$ (in particular, they are locally uniformly bounded). The assertion now follows from Theorem 1 and the observation that

$$\tilde{H}_{2n-1} = H_{2n-1} + r_{2n-1}(f'; X; \cdot). \tag{11}$$

In fact, (ii) in Theorem 1 is not possible, since otherwise $H'_{2n-1} \xrightarrow{\Lambda_1} f'$ uniformly in a neighbourhood of some point $t_0 \in \operatorname{supp}(\mu^*)$. By Rouché's theorem, for $n \in \Lambda_1$ sufficiently large, $H'_{2n-1}$ could not have more zeros (counted with multiplicities) than $f'$ in some neighbourhood of $t_0$, which contradicts the definition of $H_{2n-1}$ together with the fact that $t_0 \in \operatorname{supp}(\mu^*)$.

*Remarks.* (i) As the proof shows, the assertion of Theorem 3 can be formulated in a more general setting, e.g., for functions analytic in a neighbourhood of $[-1, 1]$. In fact, it is only needed that the classical Hermite interpolants remain bounded on $A$ and that the case (ii) of Theorem 1 with $h = f'$ is impossible. We will not dwell on the precise statement of such a generalization, but the formulation of the subsequent Theorem 4 will give a hint of what has to be done.

(ii) There are (nonconstant) $\mathscr{C}^\infty$-functions and interpolation schemes such that the corresponding Hermite–Fejér interpolants converge locally uniformly in $\mathbb{C}$ (but not necessarily on $[-1, 1]$ to the given function). For instance, one may look at $f(x) = \exp(-1/x^2)$, which has a zero of infinite multiplicity at 0, and choose an interpolation scheme that converges sufficiently fast to the Dirac measure at 0.

That there can be points outside $[-1, 1]$ in which the error function $f - H_{2n-1}$ vanishes is shown in the following simple.

EXAMPLE. Fix $\alpha \in \mathbb{R} \setminus [-1, 1]$ and consider the polynomial $f(z) := -z^3/3 + \alpha z^2/2$. Then, by (4) and (11) for $n \geqslant 2$ and any interpolation scheme $X = (x_i^{(n)})$ which is symmetric with respect to the origin,

$$f(\alpha) - H_{2n-1}(\alpha) = r_{2n-1}(f', (x_i^{(n)}), \alpha) = \sum_{i=1}^{n} x_i^{(n)} \frac{(\omega_n(\alpha))^2}{(\omega_n'(x_i^{(n)}))^2} = 0.$$

While the divergence of Hermite–Fejér interpolants outside the unit interval is studied in Theorem 3, a result on possible divergence on subintervals of $[-1, 1]$ is given in

THEOREM 4. Suppose that along some subsequence $\Lambda$, the interpolation scheme converges to some measure $\mu^*$. Assume that for some $\lambda < \lambda(\mu^*)$ (see (2)), $f$ is analytic and nonconstant in $E(\lambda, \mu^*)$. Then for each subinterval $I \subset [-1, 1]$ on which $U^{\mu^*}$ is not constantly equal to its global supremum, there holds

$$\lim_{\Lambda \ni n \to \infty} \|f - H_{2n-1}\|_I = \infty.$$

*Proof of Theorem* 4. Follows from Theorem 2 by applying the reasoning in the proof of Theorem 3. In fact, it is well known [6, p. 106] that the assumption on the analyticity of $f$ implies that the Hermite interpolants $\tilde{H}_{2n-1}$ converge to $f$ locally uniformly in $E(\lambda, \mu^*)$, which contains the interval $[-1, 1]$. ∎

COROLLARY. *Even for the locally uniform convergence on* $]-1, 1[$ *of the Hermite–Fejér interpolants to a nonconstant entire function, it is necessary that the interpolation scheme converges to the arcsine distribution.*

*Remark.* In the formulation of the Corollary we may replace a nonconstant entire function by a nonconstant function analytic in a neighbourhood of the set $G := \{z \in \mathbb{C} : \text{dist}(z, [-1, 1]) \leqslant 2\}$. In fact, it can be shown that for each unit measure $\mu^*$ on $[-1, 1]$, every neighbourhood of $G$ contains some region $E(\lambda^*, \mu^*)$ with $\lambda^* < \lambda(\mu^*)$.

Since the weak-star limit of the interpolation scheme consisting of equidistant points is the uniform distribution on $[-1, 1]$, which can be shown to have a logarithmic potential not constant on any subinterval, Theorem 4 sets the aforementioned result of Berman for $f_0(z) = z$ [1] into a new light.

We further illustrate the condition on the subintervals $I$ by the following

EXAMPLE. Consider a nondegenerate closed proper subinterval $I$ of $[-1, 1]$. From the general theory on Hermite–Fejér interpolation it follows that fr the interpolation process in the zeros of the $n$th Chebyshev polynomial of $I$,

$$\|f - H_{2n-1}\|_I \to 0$$

for every function $f$ continuous on $I$. This does not contradict the statement of Theorem 4, since the weak-star limit of the interpolation scheme is the equilibrium distribution of $I$, which has constant logarithmic potential on $I$.

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